Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures

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Abstract

We develop a reverse entropy power inequality for convex measures, which may be seen as an affine-geometric inverse of the entropy power inequality of Shannon and Stam. The specialization of this inequality to log-concave measures may be seen as a version of Milman's reverse Brunn-Minkowski inequality. The proof relies on a demonstration of new relationships between the entropy of high dimensional random vectors and the volume of convex bodies, and on a study of effective supports of convex measures, both of which are of independent interest, as well as on Milman's deep technology of M-ellipsoids and on certain information-theoretic inequalities. As a by-product, we also give a continuous analogue of some Plünnecke-Ruzsa inequalities from additive combinatorics.

1 Introduction

The reverse Brunn-Minkowski inequality is a deep result in Convex Geometry discovered by V. D. Milman in the mid 1980s (cf. [35, 36, 37, 40]). It states that, given two convex bodies A and B in $\mathbb{R}^n$, one can find linear volume preserving maps $u_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, 2)$ such that with some absolute constant C

$$\big|\widetilde{A} + \widetilde{B}\big|^{1/n} \leq C\left(|A|^{1/n} + |B|^{1/n}\right), \tag{1.1}$$

where $\widetilde{A} = u_1(A)$, $\widetilde{B} = u_2(B)$, $\widetilde{A} + \widetilde{B} = \big\{x + y : x \in \widetilde{A},\ y \in \widetilde{B}\big\}$ is the Minkowski sum, and where $|A|$ stands for the n-dimensional volume. (Of course, one of these maps may be taken to be the identity operator.) A similar inequality continues to hold for finitely many convex bodies with constants depending on the number of sets involved.

Note that the reverse inequality to (1.1),

$$\big|\widetilde{A} + \widetilde{B}\big|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{1.2}$$

holds true for any such u_i by the usual Brunn-Minkowski inequality. Without loss of generality both relations may be written for convex bodies with volume one, when (1.1)–(1.2) take a simpler form

$$2 \leq |A + \widetilde{B}|^{1/n} \leq 2C. \tag{1.3}$$

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Milman's inverse Brunn-Minkowski inequality has connections with high dimensional phenomena in Convex Geometry. For instance, it is known that proving Milman's inequality for convex bodies in isotropic position is equivalent to the hyperplane conjecture ([18]). It has also found a number of interesting extensions and applications (cf. [31], [30], [2]).

Our primary goal in this note is to develop an entropic generalization of the reverse Brunn-Minkowski inequality (1.1), which would involve arbitrary log-concave probability distributions rather than just uniform measures on compact convex sets. More generally, we consider convex (also called hyperbolic) measures, i.e., having densities of the form

$$f(x) = V(x)^{-\beta}, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where V are positive convex functions on $\mathbb{R}^n$ and $\beta > n$ is a given parameter. (To be precise, these are the densities of the so-called κ-concave measures for $\kappa = (n-\beta)^{-1}$; see Section 2 for details.) A secondary goal of this note is to develop a technology for going from entropy estimates to volume estimates in convex geometry; this is developed in Section 3, and underlies the claim that our main result, stated purely in terms of entropies, is a generalization of Milman's inverse Brunn-Minkowski inequality.

The afore-mentioned entropic generalization may be stated as an inverse of the entropy power inequality, in the same sense that Milman's inequality is an inverse of the Brunn-Minkowski inequality. Given a random vector X in $\mathbb{R}^n$ with density $f(x)$, introduce the entropy functional (or the differential entropy, or the Boltzmann–Shannon entropy),

$$h(X) = -\int_{\mathbb{R}^n} f(x) \log f(x)\, dx,$$

together with the entropy power

$$H(X) = e^{2h(X)/n},$$

provided that the integral exists in the Lebesgue sense. In particular, if X is uniformly distributed in a convex body $A \subset \mathbb{R}^n$, we have

$$h(X) = \log |A|, \qquad H(X) = |A|^{2/n}.$$

These identities themselves suggest reviewing a number of results on volume relations in terms of the entropy, and also inspire one to find analogues of such relations for different classes of multidimensional probability distributions in the language of information theory.

The entropy power inequality, due to Shannon and Stam ([47], [48], cf. also [21], [23], [53] for a refinement when one of the random vectors is normal, and [1], [33] for other refinements), asserts that

$$H(X+Y) \geq H(X) + H(Y), \tag{1.5}$$

for any two independent random vectors X and Y in $\mathbb{R}^n$, for which the entropy is defined. Although it is not directly equivalent to the Brunn-Minkowski inequality, it is very similar to it [22]. For example, being restricted to normal random vectors X, Y with covariance matrices R, S, the inequality (1.5) becomes Minkowski's inequality for determinants of positive definite matrices,

$$\det{}^{1/n}(R+S) \geq \det{}^{1/n}(R) + \det{}^{1/n}(S).$$

It includes the Brunn-Minkowski inequality for parallepipeds and therefore extends, by a simple bisection argument of Hadwiger-Ohmann (or in view of the infinitesimal character of the Brunn-Minkowski inequality), to the class of all Borel measurable subsets of the Euclidean

space. Conversely, one may deduce the entropy power inequality as a consequence of a Brunn-Minkowski inequality for restricted sums of sets [49, 50]. Moreover, both the Brunn-Minkowski and the entropy power inequalities can be given similar proofs as limiting cases of Young's inequality for convolution with sharp constant [24].

In order to judge the sharpness of the entropy power inequality (1.5), we need to keep in mind that the entropy is invariant under linear volume preserving transformation of the space, i.e., $H(u(X)) = H(X)$ whenever $|\det(u)| = 1$. On the other hand, the left side of (1.5) essentially depends on "positions" of the distributions of X and Y, in the sense that it is sensitive to linear volume preserving transformation of either X or Y. Therefore, to reverse this inequality, some transformation of these random vectors is needed. Specifically, we have:

Theorem 1.1. *Fix $\beta_0 > 2$. Let X and Y be independent random vectors in $\mathbb{R}^n$ with densities of the form (1.4) with $\beta \geq \max\{\beta_0 n, 2n+1\}$. There exist linear volume preserving maps $u_i : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$H\big(\widetilde{X} + \widetilde{Y}\big) \,\leq\, C_{\beta_0}\,(H(X) + H(Y)), \tag{1.6}$$

where $\widetilde{X} = u_1(X)$, $\widetilde{Y} = u_2(Y)$, and where C_{β_0} is a constant depending only on β_0.

For growing β, the families (1.4) shrink, and we arrive in the limit as $\beta \to +\infty$ at the class of log-concave densities (which correspond to the class of log-concave measures). Recall that log-concavity of a non-negative function f on $\mathbb{R}^n$ is also defined through the inequality

$$f(tx+sy) \geq f(x)^t g(y)^s, \qquad x,y \in \mathbb{R}^n, \ \ t,s>0, \ t+s=1.$$

Such functions are supported and positive on some open convex sets in $\mathbb{R}^n$, where $\log f$ are concave (and we define them to be zero outside supporting sets).

Thus, by Theorem 1.1, if X and Y are independent and have log-concave densities, then for some linear volume preserving maps $u_i : \mathbb{R}^n \to \mathbb{R}^n$,

$$H\big(\widetilde{X} + \widetilde{Y}\big) \,\leq\, C\,(H(X) + H(Y)), \tag{1.7}$$

where C is an absolute constant. This statement for the log-concave case was announced by the authors in [6].

As for the general case, it can be shown that there does not exist a finite universal constant such that a reverse entropy power inequality holds for the entire class of convex measures, so that some restriction on the range of convexity parameter β as in Theorem 1.1 is necessary (see Proposition 9.2). Nevertheless, it would be interesting to explore how the constants in the inequality (1.6) may depend on the remaining values $\beta > n$.

Let us state an equivalent variant of Theorem 1.1 by involving maximum of the density,

$$\|f\| = \operatorname{ess\,sup}_x f(x),$$

and keeping the same notations.

Theorem 1.2. *Fix $\beta_0 > 2$. Let X and Y be independent random vectors in $\mathbb{R}^n$ with densities f and g of the form (1.4), such that $\|f\| = \|g\| = 1$. If $\beta \geq \max\{\beta_0 n, 2n+1\}$, there exist linear volume preserving maps $u_i : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$c_0 n \,\leq\, h\big(\widetilde{X} + \widetilde{Y}\big) \,\leq\, c_{\beta_0} n \tag{1.8}$$

with some absolute constant $c_0 > 0$, and some constant c_{β_0} depending only on β_0.

Equivalently, with some $C_{\beta_0} > C_0 > 1$, we have

$$C_0 \leq H\big(\widetilde{X} + \widetilde{Y}\big) \leq C_{\beta_0}. \tag{1.9}$$

Being restricted to random vectors X and Y that are uniformly distributed in convex bodies A and B, the reverse entropy power inequality (1.6) is equivalent to Milman's theorem (1.1) modulo an absolute factor, while the right inequality in (1.9) is equivalent to the right inequality in (1.3) in a similar sense (under the assumption $|A| = |B| = 1$).

This generalization is however not immediate and has to be clarified, because the distribution of $X + Y$ is not uniform in $A + B$. Nevertheless, it is "almost" uniform, so that $H(X+Y)$ is of the same order as $|A+B|^{2/n}$. As will be explained later on, if X and Y are independent and uniformly distributed in A and B, we have

$$\frac{1}{4}\,|A+B|^{2/n} \leq\, H(X+Y) \leq\, |A+B|^{2/n}. \tag{1.10}$$

These bounds allow one to freely translate many volume relations into statements about entropy.

As for the left inequality in (1.8), it immediately follows from the entropy power inequality (1.5), which implies "concavity" of the entropy functional:

$$h\bigg(\frac{\widetilde{X} + \widetilde{Y}}{\sqrt{2}}\bigg) \geq \frac{h(\widetilde{X}) + h(\widetilde{Y})}{2} = \frac{h(X) + h(Y)}{2} \geq 0,$$

where on the last step the assumption $f, g \leq 1$ is used. Hence, one may take $c_0 = \log\sqrt{2}$ in (1.8) and $C_0 = 2$ in (1.9), similarly to the left inequality in (1.3).

It should be noted that there are other (non-entropic) formulations of the reverse Brunn-Minkowski inequality. In their study of the geometry of log-concave functions B. Klartag and V. D. Milman have recently proposed a natural functional generalization of (1.1) in terms of the Asplund product

$$f \star g(x) = \sup_y\, \big[f(x-y)g(y)\big], \qquad x \in \mathbb{R}^n.$$

They prove (cf. [30, Theorem 1.3]) that, given symmetric log-concave functions f and g on $\mathbb{R}^n$, satisfying $f(0) = g(0) = 1$, there exist linear volume preserving maps $u_i : \mathbb{R}^n \to \mathbb{R}^n$ such that with some absolute constant C,

$$\bigg(\int \tilde{f} \star \tilde{g}(x)\; dx\bigg)^{1/n} \leq\, C\,\bigg[\bigg(\int f(x)\, dx\bigg)^{1/n} + \bigg(\int g(x)\, dx\bigg)^{1/n}\bigg], \tag{1.11}$$

where $\widetilde{f}(x) = f((u_1(x))$ and $\widetilde{g}(x) = g(u_2(x))$. Indeed, on the indicator functions $f = 1_A$, $g = 1_B$, we have $\tilde{f} \star \tilde{g} = 1_{\widetilde{A} + \widetilde{B}}$, so (1.11) reduces exactly to (1.1).

The inequality (1.11) is related to the log-concave variant (1.7) in Theorem 1.1. However, the Asplund product behaves differently than the usual convolution, especially for densities that are not log-concave. Anyhow, in the proof of Theorems 1.1–1.2 themselves, the convex body case as in (1.1) or (1.3), that is, Milman's theorem, will be a basic ingredient in our argument, together with a general "submodularity" property of the entropy functional (cf. [32]), which has recently appeared in information theory.

The paper is organized as follows. In Section 2 we recall Borell's hierarchy and characterization of convex measures and discuss convexity properties of convolutions, which are prerequisites for the rest of the paper.

Section 3 introduces a new tool for going from entropy estimates to volume estimates in convex geometry. The key idea here is that for sufficiently "convex" probability measures (i.e., κ-concave probability measures for positive κ, which necessarily have compact support), the entropy can be approximated in some sense by the logarithm of the volume of the support set. While the fact that the entropy of a probability measure on a compact set is bounded from above by the logarithm of the volume of the support is simple and classical, the corresponding lower bound under convexity assumptions is new. In Section 4, the entropy of convex measures is related to the maximum of their densities (which is of course related to the volume of the support in the special case of the uniform distribution on a set), and some corollaries are discussed.

The case of negative κ is considered in Section 5. In this case, although the support set of a κ-concave probability measure may not be bounded, it is nonetheless possible to define in some sense an "effective support", which is bounded and whose volume is related to the entropy of the measure. In this sense, the relation between entropy and volume can be extended to general convex measures, and moreover, this may be thought of as providing a reverse technology to go from volume estimates to entropy estimates in convex geometry by using the notion of effective supports. Some refinements of these ideas, related to an asymptotic equipartition property for log-concave measures, are described in [13].

Next, in Section 6, we turn to the notion of M-positions of convex bodies, first developed by V. Milman, and show using the afore-mentioned effective support idea that such a notion can be defined for convex measures. Section 7 introduces into convex geometry a submodularity result for the entropy of sums, first developed in [32], and discusses some corollaries, including the connection of M-positions of convex bodies with the reverse Brunn-Minkowski inequality, and continuous analogues for volumes of convex bodies of the Plünnecke-Ruzsa inequalities that are well known in the discrete world of additive combinatorics.

Section 8 and 9 are devoted to completing the proof of Theorem 1.1– the former for the log-concave case, and the latter for the general convex measure case. Finally, in Section 10, we comment on the reverse entropy power inequality (1.7) for log-concave measures in the case where the distributions of X and Y are isotropic.

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2 Convex measures

Here we recall basic definitions and the characterization of the so-called convex measures.

Given $-\infty \leq \kappa \leq 1$, a probability measure μ on $\mathbb{R}^n$ is called κ-concave, if it satisfies the Brunn-Minkowski-type inequality

$$\mu\big(tA + (1-t)B\big) \geq \big[\, t\mu(A)^\kappa + (1-t)\mu(B)^\kappa\big]^{1/\kappa} \tag{2.1}$$

for all $t \in (0,1)$ and for all Borel measurable sets $A, B \subset \mathbb{R}^n$ with positive measure. When $\kappa = 0$, (2.1) describes the class of log-concave measures which thus satisfy

$$\mu\big(tA + (1-t)B\big) \geq \mu(A)^t \mu(B)^{1-t}.$$

In the absolutely continuous case, the log-concavity of a measure is equivalent to the log-concavity of its density (Prékopa's theorem [42]). When $\kappa = -\infty$, the right side is understood as $\min\{\mu(A), \mu(B)\}$. The inequality (2.1) is getting stronger as the parameter κ is increasing, so in the case $\kappa = -\infty$ we obtain the largest class, whose members are called convex or hyperbolic probability measures.

For general κ's, the family of κ-concave measures was introduced and studied by C. Borell [16, 17] who gave the following characterization, which we state below in the absolutely-continuous case. In this case necessarily $\kappa \leq 1/n$. See also [19].

Proposition 2.1. *An absolutely continuous probability measure μ on $\mathbb{R}^n$ is κ-concave, where $-\infty \leq \kappa \leq 1/n$, if and only if μ is supported on an open convex set $\Omega \subset \mathbb{R}^n$, where it has a positive $\tilde{\kappa}$-concave density f, that is, satisfying*

$$f(tx + (1-t)y) \geq \left[\, t f(x)^{\tilde{\kappa}} + (1-t) f(y)^{\tilde{\kappa}} \right]^{1/\tilde{\kappa}} \tag{2.2}$$

for all $t \in (0,1)$ and $x, y \in \Omega$.

Here and below we put

$$\tilde{\kappa} = \frac{\kappa}{1 - n\kappa}, \qquad \beta = \frac{1}{|\tilde{\kappa}|}.$$

Thus, μ is κ-concave if and only if f is $\tilde{\kappa}$-concave.

If $\kappa \in (0, 1/n)$, then $\tilde{\kappa} > 0$ and $\beta > 0$, and the supporting set Ω has to be bounded (so, its closure is a convex body). In this case, one may represent the density in the form $f = \varphi^\beta$, where φ is an arbitrary positive concave function on Ω, satisfying the normalization condition $\int_\Omega \varphi^\beta \, dx = 1$.

If $\kappa < 0$, then $\tilde{\kappa} < 0$ and $f = V^{-\beta}$ (like in formula (1.4)), where V is an arbitrary positive convex function on Ω, satisfying $\int_\Omega V^{-\beta} \, dx = 1$. Since $\beta = n - (1/\kappa)$ in this case, we must have $\beta > n$.

The following statement has been also well-known since the works of C. Borell, cf. e.g. [17, Theorem 4.5]. (There it is assumed additionally that $0 < \kappa', \kappa'' < 1/n$, while we will also need to consider the case when one of κ' or κ'' is negative. Nevertheless, Borell's result [17, Theorem 4.2] about κ-concavity of product measures covers the general case.)

Proposition 2.2. *Assume a probability measure μ is κ'-concave on $\mathbb{R}^n$ and a probability measure ν is κ''-concave on $\mathbb{R}^n$. If $\kappa', \kappa'' \in [-1, 1]$ satisfy*

$$\kappa' + \kappa'' > 0, \qquad \frac{1}{\kappa} = \frac{1}{\kappa'} + \frac{1}{\kappa''}, \tag{2.3}$$

then their convolution $\mu * \nu$ is κ-concave.

Taking the limit $\kappa', \kappa'' \to 0$, one also obtains the log-concavity of the convolution of any two log-concave probability measures.

The argument is based on the following elementary property of the M_κ-mean functions defined by

$$M_\kappa^{(t)}(a,b) = (ta^\kappa + sb^\kappa)^{1/\kappa}, \qquad a, b \geq 0, \quad 0 < t < 1, \; s = 1 - t,$$

with the usual meaning in the cases $\kappa = -\infty$, $\kappa = +\infty$ and $\kappa = 0$, as $\min\{a,b\}$, $\max\{a,b\}$ and $a^t b^s$, respectively. (Note these functions appear on the right sides of (2.1) and (2.2).) Namely, under the condition (2.3), for all real positive numbers a', a'', b', b'' and any $t \in (0,1)$,

$$M_{\kappa'}^{(t)}(a', b') \, M_{\kappa''}^{(t)}(a'', b'') \; \geq \; M_\kappa^{(t)}(a'a'', b'b'').$$

Consequently, if $A = A' \otimes A''$ and $B = B' \otimes B''$ with standard parallelotopes A', B' in $\mathbb{R}^n$ of positive μ-measure, and with standard parallelotopes A'', B'' in $\mathbb{R}^n$ of positive ν-measure, then

$$tA + sB = (tA' + sB') \times (tA'' + sB''),$$

and, using the definition (2.1), for the product measure $\lambda = \mu \otimes \nu$ we have:

$$\begin{aligned} \lambda(tA + sB) &= \mu(tA' + sB')\, \nu(tA'' + sB'') \\ &\geq M_{\kappa'}^{(t)}(\mu(A'), \mu(B'))\, M_{\kappa''}^{(t)}(\nu(A''), \nu(B'')) \\ &\geq M_{\kappa}^{(t)}(\mu(A')\nu(A''), \mu(B')\nu(B'')) \\ &= M_{\kappa}^{(t)}(\lambda(A),\, \lambda(B)). \end{aligned}$$

That is, the Brunn-Minkowski-type inequality (2.1) is fulfilled for the measure λ on $\mathbb{R}^{2n}$ in the class of all standard parallelotopes (of positive measure). By virtue of the standard bisection argument of Hadwiger-Ohmann [28], described, for example, in [16, 17, 20], one can extend (2.1) from the class of standard parallelopipeds to arbitrary Borel sets A and B, which means the κ-concavity of λ on $\mathbb{R}^{2n}$. Finally, since $\mu * \nu$ represents the image of λ under the linear map $(x, y) \to x + y$, the convolution is also κ-concave.

One particular case of Proposition 2.2 is the following well-known corollary:

Corollary 2.3. *If random vectors $X_1, \dots, X_m$ are independent and uniformly distributed in convex bodies $A_1, \dots, A_m$ in $\mathbb{R}^n$, then the sum*

$$X_1 + \cdots + X_m$$

has a $\frac{1}{mn}$–concave distribution supported on the convex body $A_1 + \cdots + A_m$.

3 Entropy and volume of the support

In this Section we bound the entropy of a κ-concave probability measure on $\mathbb{R}^n$ with a positive parameter of convexity κ in terms of the volume of its supporting set. Note that, for any random vector X with values in A, there is a general upper bound

$$h(X) \leq \log |A|. \tag{3.1}$$

So our concern is how to estimate the entropy from below to get

$$h(X) \geq -Cn + \log |A| \tag{3.2}$$

with constants $C \geq 0$ depending only on the "strength" of convexity of the density f of X.

To proceed, we need some preparations. Given a measurable function φ on a measurable set $A \subset \mathbb{R}^n$ and $p > 0$, write

$$\|\varphi\|_p = \Big(\int_A |\varphi|^p \, dx \Big)^{1/p}.$$

The following Khinchin-type (or reverse Hölder) inequality for the class of concave functions is due to Berwald [5] (cf. [15]).

Lemma 3.1. *Given a concave function $\varphi \geq 0$ on a convex body A in $\mathbb{R}^n$,*

$$\big(C_{n+q}^n\, |A|^{-1}\big)^{1/q}\ \ \|\varphi\|_q \leq \big(C_{n+p}^n\, |A|^{-1}\big)^{1/p}\ \ \|\varphi\|_p, \qquad 0 < p < q. \tag{3.3}$$

Here and below we use the standard binomial coefficients

$$C_q^n = \frac{q(q-1)\dots(q-n+1)}{n!}. \tag{3.4}$$

As easy to verify, the equality in (3.3) is achieved for the linear function $f(x) = x_1 + \dots + x_n$ on the convex body

$$A = \{x \in \mathbb{R}^n : x_i > 0,\ x_1 + \dots + x_n < 1\}. \tag{3.5}$$

Berwald's inequality may equivalently be stated for the class of $\tilde{\kappa}$-concave probability density functions f on A with $\tilde{\kappa} > 0$, since then $f = \varphi^{1/\tilde{\kappa}}$ with concave φ. Inserting $\varphi = f^{\tilde{\kappa}}$ into (3.3), we get

$$\left(C_{n+q}^n\, |A|^{-1}\right)^{1/q}\ \|f\|_{q\tilde{\kappa}}^{\tilde{\kappa}} \le \left(C_{n+p}^n\, |A|^{-1}\right)^{1/p}\ \|f\|_{p\tilde{\kappa}}^{\tilde{\kappa}}.$$

Choose $p = \beta = 1/\tilde{\kappa}$ so that $\|f\|_{p\tilde{\kappa}} = \|f\|_1 = 1$. The inequality is simplified (but does not lose generality):

$$\left(C_{n+q}^n\, |A|^{-1}\right)^{1/q}\ \|f\|_{q\tilde{\kappa}}^{\tilde{\kappa}} \le \left(C_{n+1/\tilde{\kappa}}^n\, |A|^{-1}\right)^{\tilde{\kappa}}.$$

Raising to the power q and then substituting $q\tilde{\kappa}$ with q, we obtain another equivalent form

$$C_{n+q\beta}^n\, |A|^{-1} \int_A f(x)^q\, dx\ \le\ \left(C_{n+\beta}^n\, |A|^{-1}\right)^q,$$

which holds true for any $q > 1$. There is equality at $q = 1$, so one may compare the derivatives. First let us take logarithms of both the sides:

$$\log C_{n+q\beta}^n + \log\, |A|^{-1} + \log \int_A f(x)^q\, dx\ \le\ q\, \log\left(C_{n+\beta}^n\, |A|^{-1}\right). \tag{3.6}$$

By the definition (3.4),

$$\frac{d}{dr}\, \log C_{n+r}^n = \sum_{i=1}^n \frac{1}{r+i}.$$

Hence, differentiating (3.6) at $q = 1$, we get

$$\sum_{i=1}^n \frac{1}{1+i/\beta} + \int_A f(x) \log f(x)\, dx\ \le\ \log\left(C_{n+\beta}^n\, |A|^{-1}\right),$$

or equivalently

$$h(X) \ge \log |A| + \sum_{i=1}^n \frac{1}{1+i/\beta} - \log C_{n+\beta}^n, \tag{3.7}$$

assuming that X has density f.

Now, let us rewrite (3.7) in terms of the convexity parameter of the distribution of X by applying the Borell characterization given in Proposition 2.1. Recall that if X has an absolutely continuous κ-concave distribution supported on A with $0 < \kappa \le 1/n$, then it has a $\tilde{\kappa}$-concave density f, where $\tilde{\kappa} = \frac{\kappa}{1-\kappa n}$.

Proposition 3.2. *Let X be a random vector in $\mathbb{R}^n$ having an absolutely continuous κ-concave distribution supported on a convex body A with $0 < \kappa \le 1/n$. Then*

$$h(X) \ge \log |A| + \sum_{i=1}^n \frac{1}{1+\tilde{\kappa} i} - \log C_{1/\kappa}^n, \tag{3.8}$$

where $\tilde{\kappa} = \frac{\kappa}{1-\kappa n}$.

For each κ, equality in (3.8) is attained for a special distribution supported on the set A defined in (3.5), with density $f(x)$ proportional to $(x_1+\dots+x_n)^{1/\tilde{\kappa}}$. For example, if $\kappa = 1/n$, then $\tilde{\kappa} = +\infty$, and X is to be uniformly distributed in A. In this case, (3.8) becomes just $h(X) \geq \log |A|$.

To simplify the bound (3.8), using again the notation $\beta = 1/\tilde{\kappa}$, we need to estimate from above the quantity

$$\log C^n_{n+\beta} - \sum_{i=1}^n \frac{\beta}{\beta+i} = \sum_{i=1}^n \Big[\log\frac{\beta+i}{i} - \frac{\beta}{\beta+i}\Big]. \tag{3.9}$$

In terms of $t = \beta/i$, the general term in the sum on the right side may be written as

$$\log\frac{\beta+i}{i} - \frac{\beta}{\beta+i} = \log(1+t) - \frac{t}{1+t},$$

which is increasing in $t \geq 0$. Hence, the function $s \rightarrow \log\frac{\beta+s}{s} - \frac{\beta}{\beta+s}$ is non-increasing. For any non-increasing continuous function $u = u(s) \geq 0$ in $s \geq 1$, one may use a general elementary bound

$$\sum_{i=1}^n u(i) \leq u(1) + \int_1^n u(s)\,ds.$$

In case of $u(s) = \log\frac{\beta+s}{s} - \frac{\beta}{\beta+s}$, we then get that the sum on the right side of (3.9) is bounded by

$$\begin{aligned}
&\big[\log(\beta+1) - \tfrac{\beta}{\beta+1}\big] + \textstyle\int_1^n \big[\log\frac{\beta+s}{s} - \frac{\beta}{\beta+s}\big]\,ds \\
&\qquad\qquad = \; n\log(\beta+n) - n\log n - \tfrac{\beta}{\beta+1} \;\leq\; n\log\tfrac{\beta+n}{n}.
\end{aligned}$$

Now, since $\beta = \frac{1}{\tilde{\kappa}} = \frac{1}{\kappa} - n$, we have $\beta + n = \frac{1}{\kappa}$ and therefore arrive at:

Corollary 3.3. *Let X be a random vector in $\mathbb{R}^n$ having an absolutely continuous κ-concave distribution supported on a convex body A with $0 < \kappa \leq 1/n$. Then*

$$h(X) \geq \log|A| + n\log(\kappa n).$$

Note when $\kappa = 1/n$, this bound is still sharp.

Now we can combine Corollaries 2.3 and 3.3 to obtain immediately:

Proposition 3.4. *If random vectors $X_1,\dots,X_m$ are independent and uniformly distributed in convex bodies $A_1,\dots,A_m$ in $\mathbb{R}^n$, then their sum $S_m = X_1+\dots+X_m$ has entropy, satisfying*

$$\log|A_1+\dots+A_m| - n\log m \;\leq\; h(S_m) \;\leq\; \log|A_1+\dots+A_m|.$$

Or, equivalently,

$$\log\Big|\frac{A_1+\dots+A_m}{m}\Big| \;\leq\; h(S_m) \;\leq\; \log|A_1+\dots+A_m|.$$

In particular, for independent random vectors X and Y in $\mathbb{R}^n$ uniformly distributed in convex bodies A and B, respectively, we always have

$$\log\Big|\frac{A+B}{2}\Big| \leq h(X+Y) \leq \log|A+B|.$$

These are exactly the inequalities in (1.10), announced in the introductory section.

4 Entropy and maximum of density

Any convex probability measure has a bounded density, i.e., the L^∞-norm $\|f\| = \sup_x f(x)$ of the density f is finite (cf. [9]). For sufficiently convex probability measures, the entropy may be related to $\|f\|$ via the following proposition, proved in [14].

Proposition 4.1. *Fix $\beta_0 > 1$. Assume a random vector X in $\mathbb{R}^n$ has a density $f = V^{-\beta}$, where V is a positive convex function on the supporting set. If $\beta \geq n+1$ and $\beta \geq \beta_0 n$, then*

$$\log \|f\|^{-1/n} \leq \frac{1}{n}\, h(X) \leq C_{\beta_0} + \log \|f\|^{-1/n}$$

with some constant C_{β_0} depending only on β_0.

The left inequality is general: It trivially holds without any convexity assumption. The right inequality is an asymptotic version of a result from [14] about extremal role of the multidimensional Pareto distributions.

Let us mention three immediate consequences of Proposition 4.1. The first is the specialization to log-concave measures.

Corollary 4.2. *If a random vector X in $\mathbb{R}^n$ has an absolutely continuous log-concave distribution with density f, then*

$$\log \|f\|^{-1/n} \leq \frac{1}{n}\, h(X) \leq 1 + \log \|f\|^{-1/n}.$$

The right inequality is attained for the n-dimensional exponential distribution (with any parameter $\lambda > 0$). This measure is concentrated on the positive orthant and has there density $f(x) = \lambda^n e^{-\lambda(x_1+\cdots+x_n)}$, $x_i > 0$.

Corollary 4.2 was observed by the first-named author around the year 2000 (motivated by relating the maximum of the density to the subgaussian norm). This observation was discussed with a few scholars but not published and consequently was not widely known. Independently, K. Ball observed this connection between $\|f\|$ and the entropy of f for centrally symmetric, log-concave densities, and publicized it in various lectures in 2003–06. He also proposed a program for approaching the hyperplane conjecture using this connection. Corollary 4.2 seems to have become well known (to experts) soon after– for example, it is implicit in the last part of the proof of Theorem 7 of Fradelizi and Meyer [26], who showed the non-symmetric extension using work of Fradelizi [25]. Unaware of parts of this history, the authors in [14] first explicitly wrote down Corollary 4.2 in the form given above.

It was observed in [14] that Corollary 4.2 can be written as a Gaussian comparison inequality. Specifically, for any log-concave density f, we have

$$-\frac{1}{2} \leq \frac{1}{n}\, h(Z) - \frac{1}{n}\, h(X) \leq \frac{1}{2}, \tag{4.1}$$

where Z is any Gaussian random vector in $\mathbb{R}^n$ with the same maximal value of the density as f. On the other hand, if we replace the assumption about the maximum with the requirement that Z has the same covariance matrix as X, one may consider a different inequality of a similar form

$$0 \leq \frac{1}{n}\, h(Z) - \frac{1}{n}\, h(X) \leq C.$$

Whether or not it is possible to choose here an absolute constant C (to serve the class of all log-concave densities) represents a question equivalent to the hyperplane conjecture (cf.

[14] for discussion, although the idea of such an equivalence should be credited to K. Ball as mentioned above). Let us also note that the dimension-free Gaussian comparison inequality (4.1) is similar in spirit to the main result of Section 3. Specifically, if for $\kappa > 0$, f is a density of a κ-concave random vector X taking values in the convex body A, and if U_A is the uniform distribution on A, (3.1)–(3.2) are equivalent to the statement

$$0 \le \frac{1}{n}\, h(\mathrm{U}_A) - \frac{1}{n}\, h(X) \le C.$$

We proceed to describe two further consequences of Proposition 4.1.

Corollary 4.3. *If random vectors X and Y in $\mathbb{R}^n$ are independent and have symmetric log-concave densities f and g, respectively, then*

$$\Big(\int f(x)g(x)\,dx\Big)^{-2/n} \le H(X+Y) \le e^2\Big(\int f(x)g(x)\,dx\Big)^{-2/n}.$$

Note that, by the symmetry assumption, the convolution $f * g(x) = \int f(x-y)g(y)\,dy$ represents a symmetric log-concave density. Hence, it attains maximum at the origin, so that

$$\|f * g\| = f * g(0) = \int f(x)g(x)\,dx.$$

Now, returning to the convex body case, let us combine Proposition 3.4 with Corollary 4.2 applied to $X = S_m$.

Corollary 4.4. *Let $X_1, \dots, X_m$ be independent and uniformly distributed in convex bodies $A_1, \dots, A_m$ in $\mathbb{R}^n$, and let f_m be the density of the sum $S_m = X_1 + \cdots + X_m$. Then*

$$1 \le \|f_m\| \cdot |A_1 + \cdots + A_m| \le (me)^n. \tag{4.2}$$

To illustrate possible implications, again assume we have two convex bodies A and B in $\mathbb{R}^n$, and let X, Y be independent and uniformly distributed in A and $-B$, respectively, that is, with densities $f(x) = \frac{1}{|A|}\,1_A(x)$, $g(x) = \frac{1}{|B|}\,1_B(-x)$. Their convolution

$$f * g(x) = \frac{1}{|A|\,|B|}\int 1_A(x-y)\,1_B(-y)\,dy = \frac{|(A-x)\cap B|}{|A|\,|B|}$$

is supported on $\Omega = A - B$, and (4.2) yields

$$\sup_x |(A-x)\cap B| \cdot |A-B| \le (2e)^n\,|A|\,|B|.$$

In fact, by a more careful application of Berwald's inequality (see [12] for details), the constant here may be slightly improved to get

$$\sup_x |(A-x)\cap B| \cdot |A-B| \le C_{2n}^n\,|A|\,|B|. \tag{4.3}$$

This inequality is known as the Rogers-Shephard inequality [44, Equation 14]. When $A = B$, and taking $x = 0$, it yields the Rogers-Shephard difference body inequality $|A - A| \le C_{2n}^n\,|A|$, with the sharp dimensional constant [43].

Note also that, since $C_{2n}^n < 4^n$, both the sides of (4.3) are of a similar order in the sense that

$$|A|^{1/n}\,|B|^{1/n} \le \sup_x |(A-x)\cap B|^{1/n}\,|A-B|^{1/n} \le 4\,|A|^{1/n}\,|B|^{1/n}. \tag{4.4}$$

Here the left inequality is just the bound $\|f * g\| \ge |\Omega|^{-1}\int f * g(x)\,dx = |A-B|^{-1}$.

In particular, for all symmetric convex bodies A and B in $\mathbb{R}^n$,

$$|A|^{1/n}\,|B|^{1/n} \le |A\cap B|^{1/n}\,|A+B|^{1/n} \le 4\,|A|^{1/n}\,|B|^{1/n}. \tag{4.5}$$

5 Essential support of convex measures

Although log-concave and more general convex measures on $\mathbb{R}^n$ do not have bounded supports, it is important to find a suitable form of Proposition 3.2 and its Corollary 3.3 which give bounds on the entropy for compactly supported convex measures. As it turns out, an "essential" part of any convex measure is supported on a certain convex body, and moreover its volume may be related to the entropy of the measure. For the class of log-concave probability measures an observation of this concentration type was first made by B. Klartag and V. D. Milman in [30], who proved the following statement (cf. [30, Corollary 2.4] or [29, Corollary 5.1]).

Proposition 5.1. *For any log-concave probability measure μ on $\mathbb{R}^n$ with density f,*

$$\mu\big\{f \geq c_0^n \, \|f\|\big\} \geq 1 - c_1^n$$

with some universal constants $c_0, c_1 \in (0,1)$.

In fact, at the expense of c_0 one may choose c_1 to be as small as we wish. See also [13] for refinements.

Our next step is to prove the following analogue of Proposition 5.1 for the class of convex measures.

Proposition 5.2. *Let μ be a probability measure on $\mathbb{R}^n$ with density $f = V^{-\beta}$, where V is a convex function on the supporting set. If $\beta \geq n+1$ and $\beta \geq \beta_0 n$ with $\beta_0 > 1$, then*

$$\mu\big\{f \geq c_0^n \, \|f\|\big\} \geq \frac{1}{2}, \tag{5.1}$$

for some $c_0 \in (0,1)$ depending on β_0, only.

At the expense of the constant c_0 the bound 1/2 on the right side of (5.1) can be replaced with any prescribed number $p \in (0,1)$. The convex body

$$K_f = \big\{x \in \mathbb{R}^n : f(x) \geq c_0^n \, \|f\|\big\}$$

may be viewed as the "$\frac{1}{2}$–support" or "essential support" of the measure μ. (The latter interpretation can be better justified by taking p to be some fixed number that is close to 1, but this is not needed for our purposes.)

Proof. By the Borell characterization theorem (Proposition 2.1), μ is supported on an open convex set Ω, where V is positive and convex. Without loss of generality, assume V attains minimum at some point $x_0 \in \Omega$, and moreover

$$V(x_0) = \min_{x \in \Omega} V(x) = 1,$$

which corresponds to $\|f\| = 1$. Introduce sublevel convex sets

$$A(\lambda) = \{x \in \Omega : f(x) > \lambda\}, \qquad 0 < \lambda < 1,$$

and similarly

$$A'(t) = \{x \in \Omega : V(x) < 1 + t\}, \qquad t > 0.$$

Thus $A(\lambda) = A'(\lambda^{-1/\beta} - 1)$. By the Brunn-Minkowski inequality, the function $\varphi(t) = |A'(t)|$ is $\frac{1}{n}$–concave in $t > 0$, that is, $\varphi(t) = \psi(t)^n$ for some concave function ψ, which is also non-negative and non-decreasing. We may assume that $\varphi(0+) = 0$ and similarly for ψ. Integrating by parts, we have

$$\int V(x)^{-\beta}\,dx = \int_0^{+\infty} (1+t)^{-\beta} d\varphi(t) = \beta \int_0^{+\infty} (1+t)^{-\beta-1} \varphi(t)\,dt,$$

that is,

$$\beta \int_0^{+\infty} (1+t)^{-\beta-1}\,\psi(t)^n\,dt = 1. \tag{5.2}$$

Fix $t_0 > 0$ and write similarly

$$\begin{aligned} 1 - \mu(A'(t_0)) \ &= \ \int_{\{V \geq 1+t_0\}} V(x)^{-\beta}\,dx \ &&= \ \int_{t_0}^{+\infty} (1+t)^{-\beta} d\varphi(t) \\ & && = \ \beta \int_{t_0}^{+\infty} (1+t)^{-\beta-1}\,\varphi(t)\,dt - (1+t_0)^{-\beta} \varphi(t_0), \end{aligned}$$

so,

$$1 - \mu(A'(t_0)) \ \leq \ \beta \int_{t_0}^{+\infty} (1+t)^{-\beta-1}\,\psi(t)^n\,dt. \tag{5.3}$$

Now, we need to estimate from above the integral (5.3) subject to (5.2). By concavity and monotonicity of ψ,

$$\psi(t) \geq \begin{cases} ct, & \text{for} \ \ 0 < t < t_0 \\ ct_0, & \text{for} \ \ t \geq t_0 \end{cases}$$

where $c = \psi(t_0)/t_0$. Hence, integrating just over the interval $(0, t_0)$, we get

$$\int_0^{+\infty} (1+t)^{-\beta-1}\,\psi(t)^n\,dt \ \geq \ c^n \int_0^{t_0} \frac{t^n}{(1+t)^{\beta+1}}\,dt \ = \ c^n \int_{s_0}^1 s^{\beta-n-1} (1-s)^n\,ds,$$

where $s_0 = 1/(1+t_0)$ and where we used the substitution $s = 1/(1+t)$. Hence, by (5.2),

$$c^n \ \leq \ \frac{1}{\beta \int_{s_0}^1 s^{\beta-n-1} (1-s)^n\,ds}. \tag{5.4}$$

On the other hand, using $\psi(t) \leq ct$, which holds for all $t > t_0$, we obtain that

$$\int_{t_0}^{+\infty} (1+t)^{-\beta-1}\,\psi(t)^n\,dt \ \leq \ c^n \int_{t_0}^{+\infty} \frac{t^n}{(1+t)^{\beta+1}}\,dt \ = \ c^n \int_0^{s_0} s^{\beta-n-1} (1-s)^n\,ds.$$

Combining (5.3) and (5.4), we get

$$1 - \mu(A'(t_0)) \ \leq \ \frac{\mathbf{P}\{\xi < s_0\}}{\mathbf{P}\{\xi > s_0\}}, \qquad s_0 = \frac{1}{1+t_0}, \tag{5.5}$$

where ξ is a random variable having the beta-distribution with parameters $(\beta - n, n+1)$, that is, with density

$$p(s) \ = \ \frac{1}{B(\beta - n, n+1)}\ s^{\beta-n-1} (1-s)^n, \qquad 0 < s < 1.$$

Now, to better understand the expression in (5.5), it is useful to relate the beta distribution to the gamma distribution. It is a well-known fact in probability that in the sense of distributions

$$\xi = \frac{\Gamma_{\beta-n}}{\Gamma_{\beta-n} + \Gamma_{n+1}},$$

where $\Gamma_{\beta-n}$ and Γ_{n+1} are independent random variables, having the gamma distribution with shape parameters $\beta - n$ and $n+1$ respectively (and with the scale parameter 1). In particular, one may write $\Gamma_{n+1} = \zeta_1 + \cdots + \zeta_{n+1}$, where the ζ_i's are independent and have a standard exponential distribution.

Note that the inequality $\xi < s_0$ is solved as $\Gamma_{n+1} > t_0\,\Gamma_{\beta-n}$. Consequently, (5.5) takes the form

$$1 - \mu(A'(t_0)) \ \leq\ \frac{\mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n}\}}{\mathbf{P}\{\Gamma_{n+1} < t_0\,\Gamma_{\beta-n}\}}. \tag{5.6}$$

Using Chebyshev's inequality, for any $\alpha > 1$ and $s \in (0,1)$, and actually with optimal $s = 1 - 1/\alpha$, one may write

$$\mathbf{P}\{\Gamma_{n+1} > \alpha(n+1)\} \leq (\mathbf{E}e^{s\zeta_1})^{n+1}\, e^{-\alpha s(n+1)} = \left(\frac{e^{-\alpha s}}{1-s}\right)^{n+1} = \left(e \cdot \alpha e^{-\alpha}\right)^{n+1}.$$

Take, for example, $\alpha = 4$, in which case the above gives

$$\mathbf{P}\{\Gamma_{n+1} > 4(n+1)\} \ \leq\ \left(\frac{4}{e^3}\right)^{n+1} \ <\ \left(\frac{1}{5}\right)^{n+1}. \tag{5.7}$$

Hence,

$$\begin{aligned} \mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n}\} \ &= \ \mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n},\ \Gamma_{n+1} > 4(n+1)\} \\ &\quad + \mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n},\ \Gamma_{n+1} < 4(n+1)\} \\ &< \ 5^{-(n+1)} + \mathbf{P}\Big\{\Gamma_{\beta-n} < \frac{4(n+1)}{t_0}\Big\}. \end{aligned}$$

In terms of $t_0 = \frac{4(n+1)}{T}$, where $T > 0$ will be choosen later on, we thus obtain that

$$\mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n}\} < 5^{-(n+1)} + \mathbf{P}\{\Gamma_{\beta-n} < T\}. \tag{5.8}$$

Now,

$$\begin{aligned} \mathbf{P}\{\Gamma_{\beta-n} < T\} \ &= \ \frac{1}{\Gamma(\beta-n)} \int_0^T x^{\beta-n-1}\, e^{-x}\, dx \\ &< \ \frac{1}{\Gamma(\beta-n)} \int_0^T x^{\beta-n-1}\, dx \ = \ \frac{T^{\beta-n}}{\Gamma(\beta-n+1)} \ = \ \frac{T^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

where we put $\alpha = \beta - n$ (which is positive). Take $T = \frac{1}{4}\,\mathbf{E}\Gamma_{\beta-n} = \frac{\alpha}{4}$, so that

$$\mathbf{P}\{\Gamma_{\beta-n} < T\} \ \leq\ \frac{(\frac{\alpha}{4})^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.9}$$

We claim that the right side of (5.9) does not exceed 1/4 for any $\alpha \geq 1$. Here we use the following observation. If ζ is a random variable with the standard exponential distribution, then $\mathbf{E}\zeta^{\alpha} = \Gamma(\alpha+1)$ and the claim takes the form

$$h(\alpha) \equiv \log \mathbf{E}\left(\frac{\zeta}{\alpha}\right)^{\alpha} \geq \log 4 - \alpha \log 4. \tag{5.10}$$

But as shown in [8], the function h is always concave on the positive half-axis $\alpha > 0$, whenever $\zeta > 0$ has a log-concave distribution. Hence, it is enough to verify (5.10) for $\alpha = 1$ and $\alpha = +\infty$. In our particular case, at the left endpoint there is equality, while Stirling's formula shows that (5.10) also holds at infinity.

Thus, $\mathbf{P}\{\Gamma_{\beta-n} < \frac{\beta-n}{4}\} \le \frac{1}{4}$ whenever $\beta - n \ge 1$, and for $t_0 = \frac{4(n+1)}{T} = \frac{16(n+1)}{\beta-n}$ the inequality (5.8) yields

$$\mathbf{P}\{\Gamma_{n+1} > t_0\,\Gamma_{\beta-n}\} < 5^{-(n+1)} + \frac{1}{4} < \frac{1}{3},$$

so that by (5.6),

$$1 - \mu\left(A'(t_0)\right) \le \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}. \tag{5.11}$$

Finally, recall that $A(\lambda) = A'(\lambda^{-1/\beta} - 1)$ or

$$A'(t_0) = A(\lambda) \quad \text{with} \quad \lambda = \left(1 + 16\,\frac{n+1}{\beta-n}\right)^{-\beta}.$$

By (5.11), for this value we have $\mu(A(\lambda)) = \mu\{f > \lambda\} \ge \frac{1}{2}$. We need an estimate of the form $\lambda \ge c^n$, with some $c > 0$ depending on β. The latter is equivalent to

$$\beta \log\left(1 + 16\,\frac{n+1}{\beta-n}\right) \le n\log C \quad (C = 1/c) \tag{5.12}$$

which is indeed fulfilled in the range $\beta \ge \beta_0 n$ with $\beta_0 > 1$ and $C = C(\beta_0)$. However, it is not true for $\beta = n + O(1)$.

To simplify (5.12), one may just use the elementary bound $\log(1+x) \le x$, so that (5.12) would follow from

$$16\beta\,\frac{n+1}{\beta-n} \le n\log C$$

which holds for all $\beta \ge \beta_0 n$ with $C = \exp\{32\beta_0/(\beta_0-1)\}$. Thus, Proposition 5.2 is proved with

$$c_0 = \exp\{-32\beta_0/(\beta_0-1)\}. \tag{5.13}$$

$\square$

Remark 5.3. A slight modification of the above argument leads to Proposition 5.1. Indeed, let f be a log-concave density such that $\|f\| = 1$. Write once more the inequality (5.6) with $t_0 = t/\beta$, $t > 0$, and recall the relation $A(\lambda) = A'(\lambda^{-1/\beta} - 1)$. Hence, (5.6) takes the form

$$1 - \mu\left(A\left(\left(1 + t/\beta\right)^{-\beta}\right)\right) \le \frac{\mathbf{P}\{\Gamma_{n+1} > t\,\Gamma_{\beta-n}/\beta\}}{\mathbf{P}\{\Gamma_{n+1} < t\,\Gamma_{\beta-n}/\beta\}}.$$

Letting $\beta \to +\infty$ and using $\Gamma_{\beta-n}/\beta \to 1$ in probability (according to the weak law of large numbers), we arrive in the limit at

$$1 - \mu\left(A(e^{-t})\right) \le \frac{\mathbf{P}\{\Gamma_{n+1} > t\}}{\mathbf{P}\{\Gamma_{n+1} < t\}}, \qquad t > 0.$$

Choose, for example, $t = 8n \ge 4(n+1)$. Then, by (5.7),

$$1 - \mu\left(A(e^{-8n})\right) < \frac{5^{-(n+1)}}{1 - 5^{-(n+1)}} < \frac{1}{5^n}.$$

Hence, Proposition 5.1 holds with $c_0 = e^{-8}$ and $c_1 = 1/5$.

Remark 5.4. By homogeneity, Propositions 5.1–5.2 may be stated for finite convex measures. In particular, if $f = V^{-\beta}$ is Lebesgue integrable, where V is a positive convex function, and $\beta \geq n+1$ and $\beta \geq \beta_0 n$ with $\beta_0 > 1$, then

$$\int f(x)\,dx \leq 2 \int_{K_f} f(x)\,dx \leq 2\,\|f\|\,|K_f|. \tag{5.14}$$

Recall that

$$K_f = \{x \in \Omega : f(x) \geq c_0^n\,\|f\|\}$$

is the essential support of μ with c_0 depending on β_0, only. (One may choose the constant (5.13)).

To illustrate how Proposition 5.2 may be applied, note that $2\,\|f\|\,|K_f| \geq 1$, according to (5.14). On the other hand, since $1 \geq \int_{K_f} f(x)\,dx \geq c_0^n \|f\|\,|K_f|$, we have that $\|f\|\,|K_f| \leq c_0^{-n}$. Thus,

$$\frac{1}{2}\,\|f\|^{-1/n} \leq \,|K_f|^{1/n} \leq c_0^{-1}\,\|f\|^{-1/n}. \tag{5.15}$$

But by Proposition 4.1 , if a random vector X has distribution μ,

$$1 \leq H(X)\,\|f\|^{2/n} \leq C$$

with constants C, depending on β, only (in case of the range as in Proposition 5.2). Hence, we arrive at:

Corollary 5.5. *Let a random vector X in $\mathbb{R}^n$ have a density $f = V^{-\beta}$, where V is a positive convex function on the supporting set. If $\beta \geq n+1$ and $\beta \geq \beta_0 n$ with $\beta_0 > 1$, then*

$$C'_{\beta_0}\,|K_f|^{2/n} \,\leq\, H(X) \,\leq\, C''_{\beta_0}\,|K_f|^{2/n},$$

where K_f is the essential support of the distribution of X, and where $C''_{\beta_0} > C'_{\beta_0} > 0$ depend on β_0, only.

6 M-position for convex bodies and measures

The so-called M-position of convex bodies was introduced by V. D. Milman in connection with reverse forms of the Brunn-Minkowski inequality, cf. [35]. By now several equivalent definitions of this important concept are known, and for our purposes we choose one of them. We refer an interested reader to the subsequent works [37], [36] and the book by G. Pisier [40], which also contains historical remarks; cf. also [11] for the relationship between M-position and isotropicity.

For any convex body A in $\mathbb{R}^n$, define

$$M(A) = \sup_{|\mathcal{E}|=|A|} \frac{|A \cap \mathcal{E}|^{1/n}}{|A|^{1/n}},$$

where the supremum is over all ellipsoids $\mathcal{E}$ with volume $|\mathcal{E}| = |A|$. The main result of V. D. Milman may be stated as follows:

Proposition 6.1. *If A is a symmetric convex body in $\mathbb{R}^n$, then with some universal constant $c > 0$*

$$M(A) \geq c. \tag{6.1}$$

By the Brunn-Minkowski and Rogers-Shephard difference body inequalities, for any convex body A in $\mathbb{R}^n$, we have $M(A) \geq \frac{1}{2}\,M(A-A)$. Hence, the symmetry assumption in (6.1) may be removed. (That this may be done was first noticed by V. Milman and A. Pajor in [39], using a different but equivalent definition of M-ellipsoids.)

If $|A \cap \mathcal{E}|^{1/n} \geq c\,|A|^{1/n}$ with a universal constant $c > 0$, then $\mathcal{E}$ is called an M-ellipsoid, or Milman's ellipsoid. It can be shown with the help of the reverse Santalo inequality due to Bourgain and Milman and using a bound such as (4.5) that, if $\mathcal{E}$ is a (symmetric) M-ellipsoid for a symmetric convex body A, then the dual ellipsoid $\mathcal{E}^{\mathrm{o}}$ is an M-ellipsoid for the dual body A^{o} (although with different absolute constants).

It follows from the definition that, for any convex body A in $\mathbb{R}^n$, one can find an affine volume preserving map $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $u(A)$ has a multiple of the unit centered Euclidean ball as an M-ellipsoid. In that case, one says that $u(A)$ is in M-position. Or equivalently, A is in M-position, if

$$|A \cap D|^{1/n} \geq c\,|A|^{1/n}, \tag{6.2}$$

where D is a Euclidean ball with center at the origin, such that $|D| = |A|$, and where $c > 0$ is universal.

The definition of an M-position may naturally be extended to the class of convex measures. Let μ be a convex probability measure on $\mathbb{R}^n$ with density f such that $\|f\| = 1$. Then we say that μ is in M-position (with constant $c > 0$), if

$$\mu(D)^{1/n} \geq c, \tag{6.3}$$

where D is a Euclidean ball with center at the origin of volume $|D| = 1$. Correspondingly, Proposition 6.1 can be generalized to a class of convex measures.

Proposition 6.2. *Let μ be a probability measure on $\mathbb{R}^n$ with density $f = V^{-\beta}$ such that $\|f\| \geq 1$, where V is a convex function on the supporting set. If $\beta \geq n+1$ and $\beta \geq \beta_0 n$ with $\beta_0 > 1$, then μ may be put in a position where*

$$\mu(D)^{1/n} \geq c_0$$

for some $c_0 \in (0,1)$ depending on β_0 (where D is the Euclidean ball of volume one).

By saying "put" we mean that, for some affine volume preserving map $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the image $u(\mu) = \mu u^{-1}$ of the measure μ under the map u is in M-position.

In particular, any log-concave probability measure μ on $\mathbb{R}^n$ with density f such that $\|f\| = 1$ may be put in M-position with a universal constant.

Proof. We may assume that $\|f\| = 1$. By Proposition 5.2, for some constant $c_0 > 0$, which only depends on β_0, the essential support of μ, i.e., the set $K_f = \{f(x) \geq c_0^n\}$ has measure $\mu(K_f) \geq 1/2$. Hence, as was already noted in Remark 5.4, we have

$$\frac{1}{2} \leq |K_f|^{1/n} \leq c_0^{-1}.$$

Put $K_f' = \frac{1}{|K_f|^{1/n}}\,K_f$, which is a convex body with volume $|K_f'| = 1$.

One may assume that K_f' contains the origin and is already in M-position (otherwise, apply to K_f' a linear, volume preserving map u to put it in M-position and consider the image $u(\mu)$ in place of μ). We claim that if K_f' is in M-position, then μ is also in M-position.

Indeed, if D is the Euclidean ball with center at the origin of volume $|D| = 1$, then (6.2) is satisfied for the set $A = K'_f$ with a universal constant $c > 0$. Since $K'_f \subset 2K_f$, we have $|K'_f \cap D| \leq |2K_f \cap D| \leq 2^n |K_f \cap D|$. Therefore,

$$\mu(D) \geq \int_{K_f \cap D} f(x)\,dx \geq c_0^n\,|K_f \cap D| \geq c_0^n \cdot 2^{-n} |K'_f \cap D| \geq \left(\frac{c_0 c}{2}\right)^n.$$

Proposition 6.2 is proved. ☐

7 Submodularity of entropy and implications

In the proof of Theorem 1.1 we apply a general submodularity property of the entropy functional, recently obtained in [32]. We state it below in the particular case of three random vectors.

Proposition 7.1. *Given independent random vectors X, Y, Z in $\mathbb{R}^n$ with absolutely continuous distributions, we have*

$$h(X + Y + Z) + h(Z) \leq h(X + Z) + h(Y + Z)$$

provided that all entropies are well-defined.

In particular, let X, Y, Z be uniformly distributed in arbitrary convex bodies A, B, D, respectively. By Proposition 3.4 with $m = 3$, we then obtain that

$$|A + B + D|^{1/n}\,|D|^{1/n} \leq\, 3\,|A + D|^{1/n}\,|B + D|^{1/n}.$$

Let us comment on the relationship between Proposition 6.1 and the reverse Brunn-Minkowski inequality from our point of view. The fact that the former implies the latter is contained in V. Milman's original papers [35, 36, 37] (cf. Pisier [40, Corollary 7.3]) and is based on arguments involving metric entropy rather than measure-theoretic entropy.

Corollary 7.2. *The existence of M-ellipsoids for symmetric, convex bodies is equivalent to the reverse Brunn-Minkowski inequality.*

Proof. Using the monotonicity of entropy, i.e., $h(X + Y + Z) \geq h(X + Y)$, we also have another variant with a somewhat better constant

$$|A + B|^{1/n}\,|D|^{1/n} \leq\, 2\,|A + D|^{1/n}\,|B + D|^{1/n}. \tag{7.1}$$

If, furthermore, all these convex bodies are symmetric and have volume one, by (4.5) applied to the couples (A, D) and (B, D), we get from (7.1) that

$$\frac{1}{|A \cap B|^{1/n}} \,\leq\, |A + B|^{1/n} \,\leq\, \frac{32}{|A \cap D|^{1/n}\,|B \cap D|^{1/n}}. \tag{7.2}$$

Therefore, if A and B are in M-position and have volume one, and D is the Euclidean ball of volume one, the right inequality in (7.2) together with the definition (6.2) of M-position leads to the reverse Brunn-Minkowski inequality in the form (1.3) with an identity linear operator,

$$|A + B|^{1/n} \,\leq\, C.$$

Note that the symmetry assumption in this conclusion can be removed by applying the above to the sets $A' = \frac{1}{|A-A|^{1/n}}\,(A-A)$ and $B' = \frac{1}{|B-B|^{1/n}}\,(B-B)$ and making use of the Rogers-Shephard difference body inequality.

The converse statement that the reverse Brunn-Minkowski inequality implies Proposition 6.1 can be based on the left side of (7.2). Indeed, let A be a symmetric convex body in $\mathbb{R}^n$ with volume one. Our hypothesis includes, in particular, that for some linear volume preserving map $u : \mathbb{R}^n \to \mathbb{R}^n$, the set $\widetilde{A} = u(A)$ satisfies

$$|\widetilde{A} + D|^{1/n} \, \leq \, C,$$

where D is the Euclidean ball of volume one, as before. But then the left inequality in (7.2) being written for the couple $(\widetilde{A}, D)$ indicates that $\widetilde{A}$ is in M-position with constant $c = 1/C$. $\square$

The following property of convex bodies in M-position is well-known. (It can be obtained, for instance, by comparing the left and right sides of inequality (7.2)). If A and B are symmetric convex bodies in M-position of volume one, then

$$|A \cap B|^{1/n} \geq c_1,$$

where $c_1 = c^2/32$ and c is Milman's constant in (6.1). If we drop the volume assumption, the above may be applied to the sets $\frac{1}{|A|^{1/n}}\,A$ and $\frac{1}{|B|^{1/n}}\,B$, which leads to the following corollary.

Corollary 7.3. *Let A and B be symmetric convex bodies in $\mathbb{R}^n$ that are in M-position. Then*

$$|A \cap B|^{1/n} \, \geq \, c_1 \, \min\{|A|^{1/n}, |B|^{1/n}\}.$$

Without the symmetry assumption, we still have a similar property

$$\sup_x |(A - x) \cap B|^{1/n} \, \geq \, c_1 \, \min\{|A|^{1/n}, |B|^{1/n}\}. \tag{7.3}$$

Indeed, by (4.4) and (7.1), the inequality (7.2) may be generalized as

$$\frac{1}{\sup_x |(A-x) \cap B|^{1/n}} \, \leq \, |A - B|^{1/n} \, \leq \, \frac{32}{|A \cap D|^{1/n}\, |B \cap D|^{1/n}},$$

where A, B, D are convex bodies of volume one and such that D is symmetric.

It was mentioned in Section 1 that we provide a technology for going from entropy to volume estimates. Let us illustrate this in the context of the submodularity phenomenon discussed here. Indeed, as described in [32, Theorem III], one consequence of submodularity is the following inequality.

Lemma 7.4. *Let X and $Y_1, \dots, Y_m$ be independent $\mathbb{R}^n$-valued random vectors with finite entropies. Let $\mathcal{C}_k$ denote the collection of all subsets of $[m] = \{1, \dots, m\}$ that are of cardinality k. Then*

$$h\Big(X + \sum_{i \in [m]} Y_i\Big) - h(X) \leq \frac{1}{\binom{m-1}{k-1}} \sum_{s \in \mathcal{C}_k} \Big[h\Big(X + \sum_{i \in s} Y_i\Big) - h(X)\Big].$$

Suppose A and $B_1, \ldots, B_m$ are compact, convex sets in $\mathbb{R}^n$ with nonempty interior, and that X is uniformly distributed on A while each Y_i is uniformly distributed on B_i. Applying Proposition 3.4, we have that

$$\begin{aligned}
\log\left[\frac{|A+\sum_{i\in[m]} B_i|}{|A|}\right] - n\log(1+m) &\leq h\Big(X + \sum_{i\in[m]} Y_i\Big) - h(X) \\
&\leq \frac{1}{\binom{m-1}{k-1}} \sum_{s\in\mathcal{C}_k} \left[h\Big(X + \sum_{i\in s} Y_i\Big) - h(X)\right] \\
&\leq \frac{1}{\binom{m-1}{k-1}} \sum_{s\in\mathcal{C}_k} \log\left[\frac{|A+\sum_{i\in s} B_i|}{|A|}\right].
\end{aligned}$$

Thus we obtain the following corollary.

Corollary 7.5. *Let $\mathcal{C}_k$ denote the collection of all subsets of $[m] = \{1, \ldots, m\}$ that are of cardinality k. Let A and $B_1, \ldots, B_m$ be convex bodies in $\mathbb{R}^n$, and suppose*

$$\Big|A + \sum_{i\in s} B_i\Big|^{\frac{1}{n}} \leq c_s |A|^{\frac{1}{n}}$$

for each $s \in \mathcal{C}_k$, with given numbers c_s. Then

$$\Big|A + \sum_{i\in[m]} B_i\Big|^{\frac{1}{n}} \leq (1+m)\Big[\prod_{s\in\mathcal{C}_k} c_s\Big]^{\frac{1}{\binom{m-1}{k-1}}} |A|^{\frac{1}{n}}.$$

In particular, by choosing $k = 1$, one already obtains an interesting inequality for volumes of Minkowski sums: for convex bodies, if $|A + B_i|^{\frac{1}{n}} \leq c_i |A|^{\frac{1}{n}}$ for each $i = 1, \ldots, m$, then

$$\Big|A + \sum_{i\in[m]} B_i\Big|^{\frac{1}{n}} \leq (1+m)\Big[\prod_{i\in[m]} c_i\Big] |A|^{\frac{1}{n}}.$$

Inequalities of this type are well known for set cardinalities in the context of *finite* subsets of groups. In fact, they are important inequalities in the field of additive combinatorics, where they are called Plünnecke-Ruzsa inequalities (see, e.g., the book of T. Tao and V. Vu [52]). These were introduced by H. Plünnecke [41] and generalized with a simpler proof by I. Ruzsa [45]; a more recent generalization is proved in [27], and entropic versions are developed in [34]. For illustration, the form of Plünnecke's inequality developed in [45] states that if $A, B_1, \ldots, B_k$ are finite sets in a commutative group and $|A| = m, |A+B_i| = \alpha_i m$, for $1 \leq i \leq k$, then there exists an $X \subset A, X \neq \phi$ such that

$$|X + B_1 + \ldots + B_k| \leq \alpha_1 \ldots \alpha_k |X|.$$

Thus one may think of Corollary 7.5 as providing continuous analogues of the Plünnecke-Ruzsa inequalities in the context of volumes of convex bodies in Euclidean spaces, where going from the discrete to the continuous incurs the extra factor of $(1+m)$, but one does not need to bother with taking subsets of the set A.

Let us note that T. Tao [51] has previously developed a continuous analogue of Freiman's theorem, which is related to the Plünnecke-Ruzsa inequalities. Specifically, [51, Proposition 7.1] asserts that if A is an open bounded non-empty subset of $\mathbb{R}^n$ such that $|A+A| \leq K|A|$

for some $K \geq 2^n$, then there exists an $\epsilon > 0$ and a set P which is the sum of $O_K(1)$ arithmetic progressions in $\mathbb{R}^n$ such that $A \subset P + B(0,\epsilon)$ and $|P + B(0,\epsilon)| \approx_K |A|$. However, this kind of continuous analogue is different in nature from the one we propose above, since it focuses on algebraic rather than convex structure. Another notable continuous analogue of Freiman's theorem is developed in the more general context of locally compact, abelian groups by T. Sanders [46].

8 The log-concave case

In the log-concave case Theorems 1.1–1.2 are somewhat simpler due to the property that the class of log-concave probability densities is closed under the convolution operation.

Let us describe the argument, assuming that X and Y have log-concave densities, say, f and g, respectively. First consider the case, where both f and g are even functions in the sense that $f(x) = f(-x)$ and $g(x) = g(-x)$.

Proof. **(of Theorem 1.1 in the symmetric log-concave case.)** In this case, the essential supports

$$K_f = \{f(x) \geq c_0^n \, \|f\|\} \quad \text{and} \quad K_g = \{g(x) \geq c_0^n \, \|g\|\},$$

where $c_0 \in (0,1)$ is a universal constant, are symmetric convex sets. By Corollary 4.3 , one may bound the entropy power as follows:

$$\begin{aligned} H(X+Y) \ &\leq \ e^2 \Big(\int f(x) g(x) \, dx \Big)^{-2/n} \\ &\leq \ e^2 c_0^{-4} \, \|f\|^{-2/n} \|g\|^{-2/n} \, |K_f \cap K_g|^{-2/n}. \end{aligned}$$

Moreover, if both K_f and K_g are in M-position, which may be assumed, then we have by deploying Corollary 7.3 and relation (5.15) that

$$|K_f \cap K_g|^{1/n} \geq c_1 \min\{|K_f|^{1/n}, |K_g|^{1/n}\} \geq \frac{c_1}{2} \, \min\Big\{\|f\|^{-1/n}, \|g\|^{-1/n}\Big\}.$$

Hence, with some numerical constant $C > 0$

$$H(X+Y) \leq C \, \max\Big\{\|f\|^{-2/n}, \|g\|^{-2/n}\Big\} \leq C \, \max\{H(X), H(Y)\},$$

where on the last step we made use of the general relation $H(X) \geq \|f\|^{-2/n}$. This proves Theorem 1.1 (and therefore Theorem 1.2) in the symmetric log-concave case. $\square$

In the general non-symmetric case one may use the inequality (7.3) for non-symmetric sets in M-position. There is also another argument based on the following elementary observation.

Lemma 8.1. *For any log-concave probability density f on $\mathbb{R}^n$,*

$$2^{-n} \, \|f\| \ \leq \ \int f(x)^2 \, dx \ \leq \ \|f\|. \tag{8.1}$$

The right inequality is trivial and holds without any assumption on the density. To derive the left inequality, write the definition of the log-concavity,

$$f(tx + sy) \geq f(x)^t \, f(y)^s, \quad x, y \in \mathbb{R}^n, \ t, s > 0, t + s = 1.$$

It may also be applied to $f^{1/t}$, so $f(tx+sy)^{1/t} \geq f(x)\, f(y)^{s/t}$. Integrating with respect to x and using the assumption that $\int f = 1$, we get

$$t^{-n} \int f(x)^{1/t}\, dx \geq f(y)^{s/t}.$$

It remains to optimize over y's, so that $\int f(x)^{1/t}\, dx \geq t^n\, \|f\|^{s/t}$, and then take the values $t = s = 1/2$.

Proof. **(of Theorem 1.1 in the general log-concave case.)** One may use symmetrization. Let X be a random vector in $\mathbb{R}^n$ with a log-concave density f. Let X' be an independent copy of X, thus with density $\tilde{f}(x) = f(-x)$. Then the random vector $X'' = X - X'$ has a symmetric log-concave distribution with density

$$f * \tilde{f}(x) = \int f(x+y) f(y)\, dy,$$

whose norm satisfies, by (8.1),

$$\|f\| \geq \|f * \tilde{f}\| = f * \tilde{f}(0) \geq 2^{-n}\, \|f\|. \tag{8.2}$$

Now, let's do the same symmetrization with another log-concave random vector Y in $\mathbb{R}^n$ with density g, assuming that it is independent of X. Then we are in position to apply to (X'', Y'') the symmetric part of Theorem 1.1, which gives

$$H(u_1(X'') + u_2(Y'')) \leq C\,(H(X'') + H(Y'')), \tag{8.3}$$

for some linear volume preserving map $u_i : \mathbb{R}^n \to \mathbb{R}^n$ and some universal constant C.

But since the entropy power may only increase when adding to a given random vector an independent summand, the left side of (8.3) is greater than or equal to $H(u_1(X) + u_2(Y))$. On the other hand, by Corollary 4.2 and applying (8.2), we have

$$H(X'') \leq e^2\, \big\|f * \tilde{f}\big\|^{-2/n} \leq 2e^2\, \|f\|^{-2/n} \leq 2e^2\, H(X).$$

With a similar bound for the random vector Y, we arrive at

$$H(u_1(X) + u_2(Y)) \leq 2e^2 C\,(H(X) + H(Y)).$$

□

9 Proof of Theorem 1.1

In order to involve in Theorem 1.1 more general convex measures, we need to apply the more delicate Propositions 4.1, 5.2 and 6.2. Moreover, since the previous argument based on the log-concavity of the convolution of two log-concave densities has no extension to the class of convex measures (with negative convexity parameter κ), we have to appeal to the submodularity property of the entropy functional.

Throughout this section let Z denote a random vector in $\mathbb{R}^n$ uniformly distributed in the Euclidean ball D with center at the origin and volume one. In particular, $h(Z) = 0$, and by Proposition 7.1,

$$h(X+Y) \leq h(X+Z) \,+\, h(Y+Z), \tag{9.1}$$

for all random vectors X and Y in $\mathbb{R}^n$ that are independent of each other and of Z (provided that all entropy powers are well-defined).

Let X and Y have densities of the form (1.4). In view of the homogeneity of the inequality (1.6) of Theorem 1.1, we may assume that $\|f\| \geq 1$ and $\|g\| \geq 1$. Then, by (9.1), our task reduces to showing that both $h(X+Z)$ and $h(Y+Z)$ can be bounded from above by quantities, depending on β_0, only (under further assumption on β_0). This can be achieved by putting the distributions of X and Y in M-position.

Thus, what we need is:

Lemma 9.1. *Let X be a random vector in $\mathbb{R}^n$ independent of Z with density $f = V^{-\beta}$ such that $\|f\| \geq 1$, where V is a convex function, and where β is in the range*

$$\beta \geq \max\{2n+1, \beta_0 n\} \ (\beta_0 > 2). \tag{9.2}$$

Then for some linear volume preserving map $u : \mathbb{R}^n \to \mathbb{R}^n$, we have $H(u(X)+Z) \leq C_{\beta_0}$ with constants depending on β_0, only.

Proof. By Proposition 6.2, for some affine volume preserving map $u : \mathbb{R}^n \to \mathbb{R}^n$, the distribution $\widetilde{\mu}$ of $\widetilde{X} = u(X)$ satisfies

$$\widetilde{\mu}(D)^{1/n} \geq c_0$$

with a numerical constant $c_0 > 0$ (which does not depend on β_0, since β_0 is well separated from 1). Let $\tilde{f}$ denote the density of $\widetilde{X} = u(X)$. Then the density p of $S = \widetilde{X} + Z$, given by

$$p(x) = \int_D \tilde{f}(x-z)\,dz = \widetilde{\mu}(D-x),$$

satisfies

$$\|p\| \geq p(0) \geq c_0^n. \tag{9.3}$$

Hence, in order to bound the entropy power $H(S)$, it will be sufficient to know the convexity parameter of the distribution of S. (Here is the place where the conditions (9.2) arise).

As we know from the Borell characterization, the distribution $\widetilde{\mu}$ of $\widetilde{X}$ is κ'-concave with the convexity parameter

$$\kappa' = -\frac{1}{\beta - n}.$$

Also, recall that Z has the uniform distribution in D with the parameter $\kappa'' = \frac{1}{n}$. In order to judge about convexity properties of the convolution $p = \widetilde{f} * g$, where $g = 1_D$ is the density of Z, one may apply Proposition 2.2. Then we need to check the condition

$$\kappa' + \kappa'' > 0,$$

which in our case is equivalent to $\beta > 2n$. By (9.2), this requirement is met, so S has a κ-concave distribution with parameter κ given by

$$\frac{1}{\kappa} = \frac{1}{\kappa'} + \frac{1}{\kappa''} = -(\beta - 2n),$$

that is, with $\kappa = -\frac{1}{\beta - 2n}$. Equivalently, S has a density of the form $p = W^{-\beta_S}$ for some convex function W and with the β-parameter

$$\beta_S = n - \frac{1}{\kappa} = n + (\beta - 2n) = \beta - n.$$

We can now apply Proposition 4.1 to the random vector S. Together with (9.3) it gives

$$H(S) \leq C \, \|p\|^{-2/n} \leq C \cdot c_0^{-2},$$

provided that $\beta_S \geq n+1$, $\beta_S \geq \beta_0' \, n$, $\beta_0' > 1$, and with constants depending on β_0'. With $\beta_0' = \beta_0 - 1$, these conditions are equivalent to (9.2).

Lemma 9.1 and therefore Theorem 1.1 are proved. □

It would be interesting to explore the range of β, such that the inequality of Theorem 1.1 holds true with β-dependent constants. On the other hand, the following statement (proved in [7]) is true:

Proposition 9.2. *For any constant C, there is a convex probability measure μ on the real line with the following property. If X and Y are independent random variables distributed according to μ, then* $\min(H(X+Y), H(X-Y)) \geq C H(X)$.

In other words, Theorem 1.1 does not hold with an absolute constant to serve for the entire class of convex measures (already in dimension one).

10 Discussion

One may wonder how to find specific positions (that is, the linear maps u_1 and u_2) for the distributions of the random vectors X and Y in Theorem 1.1. Natural candidates are the so-called isotropic positions.

Let us recall the well-known and elementary fact that, in the class of all (absolutely continuous) probability distributions on $\mathbb{R}^n$ with a fixed covariance matrix, the entropy $h(X)$ is maximized when X has a normal distribution. Equivalently, for any affine volume preserving map T of the space $\mathbb{R}^n$,

$$\frac{1}{2\pi e} \, H(X) \leq \int \frac{|Tx|^2}{n} \, f(x) \, dx, \tag{10.1}$$

where f is density of X. If the right side of (10.1) is minimized for the identity map $T(x) = x$, then one says that the distribution of X is isotropic or in isotropic position (cf. [38]). This is equivalent to the property that X has mean at the origin and, for any unit vector θ,

$$L_f^2 = \|f\|^{2/n} \int \langle x, \theta \rangle^2 \, f(x) \, dx,$$

for some number $L_f > 0$, called the *isotropic constant* of f. If X is uniformly distributed in a convex body K, the number $L_f = L_K$ is called the isotropic constant of K.

Thus, for any random vector X in $\mathbb{R}^n$ with density f regardless of whether its distribution is isotropic or not, (10.1) may be rewritten as

$$\frac{1}{2\pi e} \, H(X) \leq L_f^2 \, \|f\|^{-2/n}. \tag{10.2}$$

In view of the general bound $H(X) \geq \|f\|^{-2/n}$, the above estimate implies, in particular, that $L_f^2 \geq \frac{1}{2\pi e}$, so the isotropic constants are separated from zero.

Restricting ourselves to (isotropic) log-concave probability distributions, the question of whether the isotropic constants are bounded from above by a dimension-free constant is equivalent to the (still open) hyperplane problem raised by J. Bourgain in the mid 1980's. As

was shown by K. Ball [4], it does not matter whether this problem is stated for the class of (all) convex bodies or for the class of (all) log-concave distributions; see also [10] for an extension to the class of convex measures. An affirmative solution of the hyperplane problem is known for some subclasses of log-concave distributions. For example, L_f is bounded by a universal constant, if the distribution of X is log-concave and symmetric about the coordinates axes.

Anyhow, the inequalities (10.1)–(10.2) suggest the following variant of the reverse Brunn-Minkowski inequality. Let X and Y be independent random vectors with log-concave densities f and g, respectively. Applying (10.1) to $X+Y$ with $Tx = x - x_0$, where $x_0 = \mathbf{E}\,(X+Y)$, we obtain that

$$\frac{1}{2\pi e}\,H(X+Y)\,\leq\,\frac{1}{n}\int|x|^2\,f(x)\,dx+\frac{1}{n}\int|x|^2\,g(x)dx. \tag{10.3}$$

Here the right side is sharpened, when the distributions of X and Y are put in the isotropic position, and then we arrive at

$$\frac{1}{2\pi e}\,H(\widetilde{X}+\widetilde{Y})\,\leq\,L_f^2\,H(X)+L_g^2\,H(Y), \tag{10.4}$$

where $\widetilde{X} = u_1(X)$, $\widetilde{Y} = u_2(Y)$, and where affine volume preserving maps u_i's are chosen so that both $\widetilde{X}$ and $\widetilde{Y}$ are isotropic. (Such maps are easily described in terms of the covariance matrices of X and Y).

In particular, if X and Y are uniformly distributed in convex bodies A and B, respectively, the inequalities (10.3)–(10.4) together with the lower bound in (1.10) yield

$$\frac{1}{8\pi e}\,|A+B|^{2/n}\,\leq\,\frac{1}{n|A|}\int_A|x|^2\,dx+\frac{1}{n|B|}\int_B|x|^2\,dx.$$

In particular, one obtains the following corollary.

Corollary 10.1. *Suppose A and B are convex bodies, and $\widetilde{A} = u_1(A)$ and $\widetilde{B} = u_2(B)$ are the bodies after being put in isotropic position. Then*

$$\frac{1}{8\pi e}\,\big|\widetilde{A}+\widetilde{B}\big|^{2/n}\,\leq\,L_A^2\,|A|^{2/n}+L_B^2\,|B|^{2/n}.$$

Therefore, if the isotropic constants L_A and L_B are known to be bounded by a constant, say C_0, then Corollary 10.1 provides a reverse Brunn-Minkowski inequality (1.1) with $C = C_0\sqrt{8\pi e}$.

A result such as Corollary 10.1 was first obtained, using a different argument, by K. Ball in his thesis [3].

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